

# Some phenomena in basic differential calculus

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## 1 On the differentiability of inverse functions

The theorem on the differentiability of the inverse function can be formulated in the following way [3].

**Theorem 1** *Let  $I \subseteq \mathbb{R}$  be some open interval,  $x_0 \in I$  and  $f : I \rightarrow \mathbb{R}$  be a function such that*

- (i)  *$f$  is differentiable at  $x_0$  with  $f'(x_0) \neq 0$  and*
- (ii)  *$f$  is one-to-one and the inverse  $f^{-1} : f(I) \rightarrow I$  is continuous at  $y_0 := f(x_0)$ .*

*Then  $f^{-1}$  is differentiable at  $y_0$ , and its derivative is*

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

A more convenient though less general version of this theorem would be to replace condition (ii) by the following condition

- (ii)'  *$f$  is one-to-one and continuous on  $I$  (i.e. it is strictly monotonic).*

Of course, (ii)' implies condition (ii) since the inverse function of a continuous strictly monotonic function is continuous.

The condition that  $f$  is one-to-one in (ii) resp. (ii)' cannot be skipped since the condition  $f'(x_0) \neq 0$  does not imply that  $f$  is monotonic in some neighbourhood of  $x_0$  (even if  $f$  is differentiable on the whole of  $I$ ). This is illustrated by the function [2, p. 37]

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) := \begin{cases} x + \alpha x^2 \sin \frac{1}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases} \quad \text{where } \alpha \neq 0.$$

which is differentiable everywhere with  $f'(0) = 1 \neq 0$ , but is not one-to-one on any neighbourhood of 0 since  $f'$  assumes both positive and negative values there. In fact,  $f'$  is unbounded in both directions on each such neighbourhood (see Figure 1).

However, if in Theorem 1 one assumes that  $f$  is differentiable on  $I$  with  $f'(x) \neq 0$  for all  $x \in I$  (not just for  $x = x_0$ ), then  $f$  is one-to-one by Darboux's intermediate value theorem.

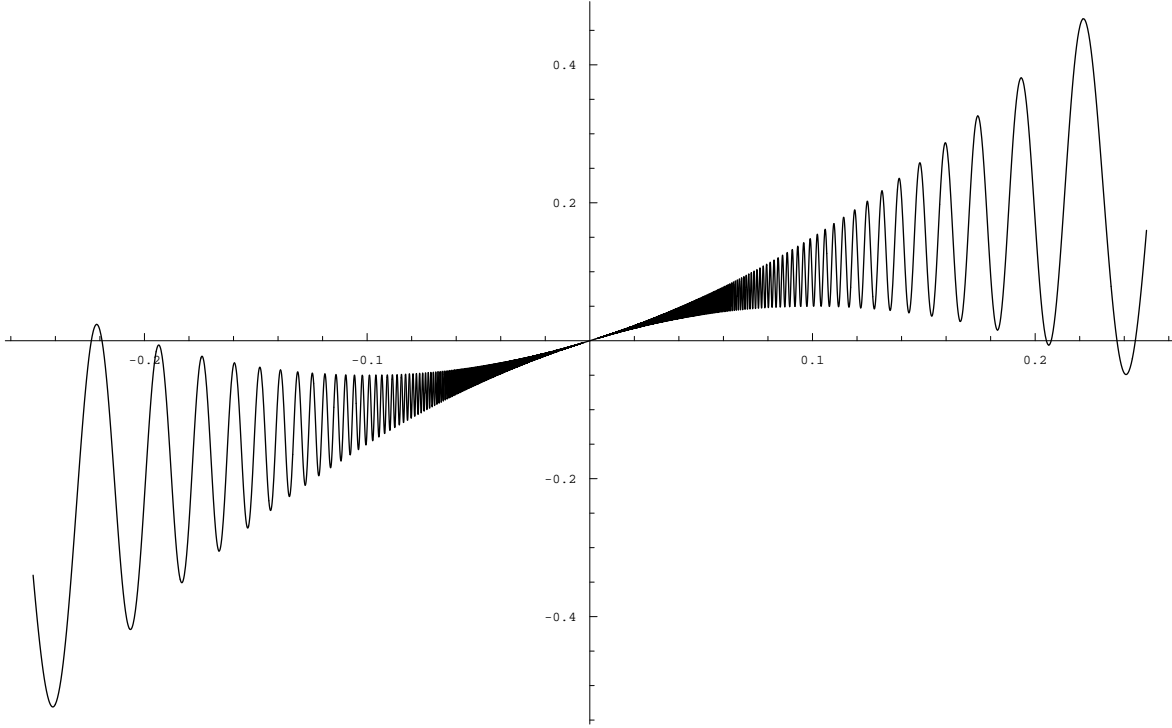


Figure 1:  $f(x) = x + 5x^2 \sin \frac{1}{x}$

On the other hand, it does not suffice either just to assume (in (ii) resp. (ii)') that  $f$  is one-to-one (as it is done, for example, in [1] and [4]); the Theorem might fail if  $f^{-1}$  is not continuous at  $y_0$ . This is illustrated by the following example.

**Example 2** We set  $I := [-2, 2]$  and

$$A := [-2, 2] \setminus \left( \{0\} \cup \bigcup_{n=1}^{\infty} \left[ \frac{1}{n+1}, \frac{2n+1}{2n(n+1)} \left[ \cup \bigcup_{n=1}^{\infty} \right] - \frac{2n+1}{2n(n+1)}, -\frac{1}{n+1} \right] \right).$$

(Observe that  $\frac{2n+1}{2n(n+1)}$  is just the middle of the interval  $[\frac{1}{n+1}, \frac{1}{n}]$ .)

Then  $A$  is an uncountable subset of  $\mathbb{R}$ , so by the Cantor-Bernstein-Schroeder theorem there exists a one-to-one map  $T$  of  $[-2, -1] \cup [1, 2]$  onto  $A$ . We define  $f : [-2, 2] \rightarrow \mathbb{R}$  by

$$f(x) := \begin{cases} 0 & \text{for } x = 0, \\ \frac{1}{n+1} + \frac{1}{2} \left( x - \frac{1}{n+1} \right) & \text{for } \frac{1}{n+1} \leq x < \frac{1}{n}, \\ -\frac{1}{n+1} + \frac{1}{2} \left( x + \frac{1}{n+1} \right) & \text{for } -\frac{1}{n} < x \leq -\frac{1}{n+1}, \\ T(x) & \text{for } x \in [-2, -1] \cup [1, 2]. \end{cases}$$

The graph of  $f$  in the interval  $[0.05, 0.6]$  is sketched in Figure 2. It is easy to see that  $f(-x) = -f(x)$  for  $-1 < x < 1$ .

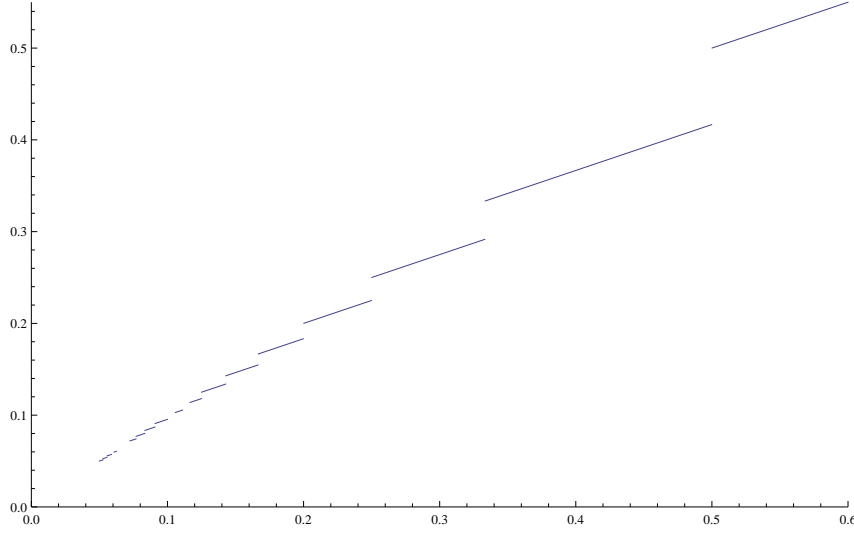


Figure 2: The graph of the function in Example 2 in the interval  $[0.05, 0.6]$

Since  $f$  maps the intervals  $[\frac{1}{n+1}, \frac{1}{n}[$  linearly onto  $[\frac{1}{n+1}, \frac{2n+1}{2n(n+1)}[$ , it is clear that  $f$  maps  $I$  onto itself in a one-to-one fashion. For any given  $x \in [\frac{1}{n+1}, \frac{1}{n}[$  we can write  $x = \frac{1}{n+1} + \tau$  with  $0 \leq \tau < \frac{1}{n(n+1)}$ , so we obtain

$$\frac{f(x) - f(0)}{x - 0} = \frac{\frac{1}{n+1} + \frac{\tau}{2}}{\frac{1}{n+1} + \tau} = \frac{1 + \frac{\tau}{2}(n+1)}{1 + \tau(n+1)} \begin{cases} \leq 1, \\ \geq \frac{1}{1+\frac{1}{n}}. \end{cases}$$

From this (and the fact that  $f$  is an odd function on  $] -1, 1[$ ) we see that  $f'(0) = 1$ . However,  $f^{-1}$  is not continuous at  $f(0) = 0$  since  $f^{-1}(A) = [-2, -1] \cup [1, 2]$  and since 0 is an accumulation point of  $A$ .  $\square$

## 2 Reparametrizations of smooth curves

**Definition 1** We say that a curve, i.e. a continuous mapping  $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ , is **smooth** if it is continuously differentiable and one-to-one<sup>1</sup> and if  $\gamma'(t) \neq 0$  for every  $t \in [a, b]$ . We say that it is **piecewise smooth** if there exists a partition  $a = t_0 < t_1 < \dots < t_k = b$  of  $[a, b]$  such that the restrictions  $\gamma|_{[t_{j-1}, t_j]}$  ( $j = 1, \dots, k$ ) are smooth.

Let  $\gamma_1 : [a, b] \longrightarrow \mathbb{R}^n$  and  $\gamma_2 : [c, d] \longrightarrow \mathbb{R}^n$  be two curves in  $\mathbb{R}^n$ . We say that  $\gamma_1$  is attained from  $\gamma_2$  by **regular reparametrization** or that  $\gamma_1$  is a regular reparametrization of  $\gamma_2$

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<sup>1</sup>The definition of smooth curves is not unique in the literature. Sometimes it is required that they are one-to-one, sometimes not.

if there exists a continuously differentiable and bijective function  $\varphi : [a, b] \longrightarrow [c, d]$  with  $\varphi'(t) \neq 0$  for all  $t \in [a, b]$  such that  $\gamma_1 = \gamma_2 \circ \varphi$ . We call  $\varphi$  the corresponding **parameter transformation**. Furthermore, in this situation we say that  $\gamma_1$  and  $\gamma_2$  are **equivalent**.

This is an equivalence relation on the set of curves, the symmetry being a consequence of Theorem 1.

The following result seems to be almost a matter of course, but its proof is surprisingly intricate. We couldn't find it in the literature.

**Theorem 3** *Let  $\gamma = (\gamma_1, \dots, \gamma_n) : [a, b] \longrightarrow \mathbb{R}^n$  and  $\eta = (\eta_1, \dots, \eta_n) : [c, d] \longrightarrow \mathbb{R}^n$  be two smooth curves with the same trace  $\Gamma := \gamma([a, b]) = \eta([c, d])$ . Then  $\gamma$  is attained from  $\eta$  by a regular reparametrization.*

**Proof.** If  $n = 1$ , this follows immediately from Theorem 1. Therefore, we may assume  $n \geq 2$ .

Of course, since  $\gamma$  and  $\eta$  are one-to-one, the only candidate for a regular parameter transformation  $\varphi$  such that  $\gamma = \eta \circ \varphi$  is

$$\varphi := \eta^{-1} \circ \gamma.$$

We have to show that  $\varphi$  is continuously differentiable with  $\varphi'(t) \neq 0$  for all  $t$ . So let some  $t_0 \in [a, b]$  be given. Then there is a unique  $s_0 \in [c, d]$  such that  $\gamma(t_0) = \eta(s_0)$ . Also, for every  $h$  with  $|h|$  small enough<sup>2</sup>, there is a unique  $\lambda(h)$  such that

$$\eta(s_0 + h) = \gamma(t_0 + \lambda(h)).$$

In view of the compactness of the intervals  $[a, b]$  and  $[c, d]$ ,  $\gamma : [a, b] \longrightarrow \Gamma$  and  $\eta : [c, d] \longrightarrow \Gamma$  are homeomorphisms. In particular,  $\gamma^{-1}$  and  $\eta^{-1}$  are continuous, so we can conclude that

$$h \longrightarrow 0 \quad \Longleftrightarrow \quad \lambda(h) \longrightarrow 0.$$

Since  $\eta$  is smooth, w.l.o.g. we can assume that  $\eta'_1(s_0) \neq 0$ .

**Claim 1:**  $\gamma'_1(t_0) \neq 0$ .

**Proof of Claim 1.** For  $h$  small enough, we have

$$\frac{\gamma_1(t_0 + \lambda(h)) - \gamma_1(t_0)}{\lambda(h)} = \frac{\eta_1(s_0 + h) - \eta_1(s_0)}{h} \cdot \frac{h}{\lambda(h)}.$$

For  $h \rightarrow 0$ , this yields

$$\gamma'_1(t_0) = \eta'_1(s_0) \cdot \lim_{h \rightarrow 0} \frac{h}{\lambda(h)}.$$

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<sup>2</sup>Here, of course, for  $s_0 = c$  (resp.  $s_0 = d$ ) only  $h > 0$  (resp.  $h < 0$ ) are admissible.

If by negation  $\gamma'_1(t_0) = 0$ , then we would have  $\lim_{h \rightarrow 0} \frac{h}{\lambda(h)} = 0$ . But since  $\gamma$  is smooth, there exists a  $j \in \{2, \dots, n\}$  such that  $\gamma'_j(t_0) \neq 0$ . We may assume that  $j = 2$ , i.e.  $\gamma'_2(t_0) \neq 0$ . Then in

$$\frac{\gamma_2(t_0 + \lambda(h)) - \gamma_2(t_0)}{\lambda(h)} = \frac{\eta_2(s_0 + h) - \eta_2(s_0)}{h} \cdot \frac{h}{\lambda(h)}$$

the left hand side tends to  $\gamma'_2(t_0) \neq 0$  if  $h \rightarrow 0$  while the right hand side tends to  $\eta'_2(s_0) \cdot 0 = 0$ , a contradiction. This proves Claim 1.

**Claim 2:** For  $t$  close enough to  $t_0$  and for  $s$  close enough to  $s_0$  we have

$$\gamma(t) = \eta(s) \quad \Longleftrightarrow \quad \gamma_1(t) = \eta_1(s).$$

**Proof of Claim 2.** “ $\implies$ ” is obvious.

To prove “ $\impliedby$ ”, we assume to the contrary that there are sequences  $(t_n), (s_n)_n$  such that  $\lim_{n \rightarrow \infty} t_n = t_0$ ,  $\lim_{n \rightarrow \infty} s_n = s_0$  and  $\gamma_1(t_n) = \eta_1(s_n)$ , but  $\gamma(t_n) \neq \eta(s_n)$  for all  $n$ . Then there are  $s_n^* \neq s_n$  such that  $\lim_{n \rightarrow \infty} s_n^* = s_0$  and  $\gamma(t_n) = \eta(s_n^*)$  for all  $n$ . So we have  $\eta_1(s_n) = \eta_1(s_n^*)$  for all  $n$ , and by Rolle's theorem we deduce the existence of points  $\xi_n$  such that  $\eta'_1(\xi_n) = 0$  and  $\lim_{n \rightarrow \infty} \xi_n = s_0$ . For  $n \rightarrow \infty$  we obtain  $\eta'_1(s_0) = 0$  since  $\eta'_1$  is continuous. This is a contradiction.

Now since  $\eta'_1(s_0) \neq 0$  and  $\eta'_1$  is continuous, the inverse  $\eta_1^{-1}$  exists in some neighbourhood of  $\eta_1(s_0)$ , and from Claim 2 we see that

$$\varphi = \eta^{-1} \circ \gamma = \eta_1^{-1} \circ \gamma_1$$

in some neighbourhood  $U_0$  of  $t_0$ . By Theorem 1,  $\eta_1^{-1}$  and hence  $\varphi$  is continuously differentiable and

$$\varphi'(t) = (\eta_1^{-1})'(\gamma_1(t)) \cdot \gamma'_1(t) = \frac{\gamma'_1(t)}{\eta'_1(\eta_1^{-1}(\gamma_1(t)))}$$

for all  $t \in U_0$ . By Claim 1 and the continuity of  $\gamma'_1$  and  $\eta'_1$  we have  $\varphi'(t) \neq 0$  for all  $t$  in a neighbourhood  $U \subseteq U_0$  of  $t_0$ , and  $\varphi$  is continuous on  $U$ .

Since this is true in a certain neighbourhood of any given  $t_0 \in [a, b]$ , the proof of Theorem 3 is completed. ■

**Remark:** If in Theorem 3 we only assume that  $\gamma$  and  $\eta$  are *piecewise* smooth, then the proofs shows that at least there is a piecewise smooth parameter transformation between  $\gamma$  and  $\eta$ .

Of course, the condition that  $\gamma$  and  $\eta$  are one-to-one cannot be skipped as the two curves

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2, \gamma(t) := (\cos t, \sin t) \quad \text{and} \quad \eta : [0, 3\pi] \longrightarrow \mathbb{R}^2, \eta(t) := (\cos t, \sin t)$$

which have the same trace but are not equivalent demonstrate.

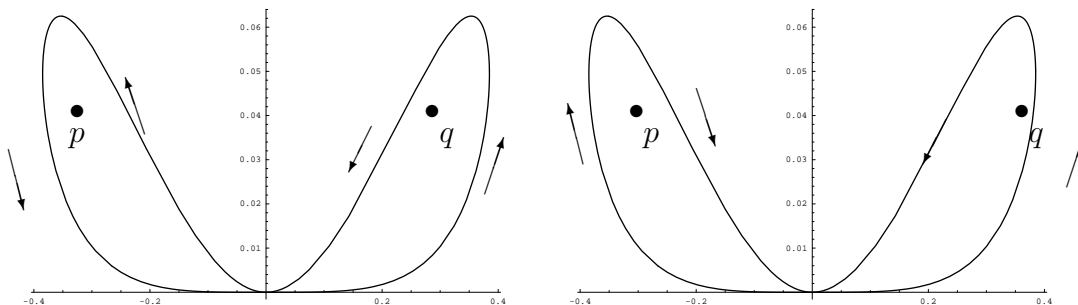


Figure 3: Two curves which are not equivalent, but have the same trace

Figure 3 shows the common trace of two  $C^1$ -curves  $\gamma$  and  $\eta$  in  $\mathbb{R}^2$ , both starting from the origin (and first “turning to the right”) which are both “almost” one-to-one (except for the origin which is attained three times) and which even attain each value in their common trace the same number of times. However, they are not equivalent. This can be seen as follows: We can assume that  $\gamma, \eta : [0, 2] \rightarrow \mathbb{R}^2$  have the same parameter interval  $[0, 2]$  and that both  $\gamma$  and  $\eta$  map  $[0, 1]$  onto the right loop in the picture, around  $q$ , and in the same direction, while  $\gamma$  maps  $[1, 2]$  onto the left loop around  $p$  clockwise and  $\eta$  does the same counterclockwise. If there would exist a regular reparametrization  $\varphi$  between  $\gamma$  and  $\eta$ , it would map both  $[0, 1]$  onto itself and  $[1, 2]$  onto itself, but it would be increasing (i.e. preserving the orientation) on  $[0, 1]$  and decreasing (altering the orientation) on  $[1, 2]$ . Hence it cannot be continuous at 1. This is a contradiction.

### 3 Some more things you can do with $x \mapsto \sin \frac{1}{x}$

Functions of the kind  $x \mapsto x^k \sin \frac{1}{x^m}$  are popular counterexamples in basic calculus. For instance, the function  $x \mapsto x^2 \sin \frac{1}{x^2}$  shows that derivatives are not necessarily continuous, that they can even be unbounded on compact intervals, and that differentiable functions need not be rectifiable on compact intervals, i.e. their graph can have infinite length. (From the point of view of complex analysis, of course the interesting properties of these functions are related to the fact that  $z \mapsto z^m \sin \frac{1}{z}$  has an essential singularity at  $z = 0$ .)

Here are some further “applications” of functions of this kind:

1. If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuously differentiable function which has a strict local minimum at some point  $x_0$ , then most students tend to expect that there is a small neighbourhood  $]x_0 - \delta, x_0 + \delta[$  such that  $f$  is decreasing on  $]x_0 - \delta, x_0]$  and increasing on  $[x_0, x_0 + \delta[$ .

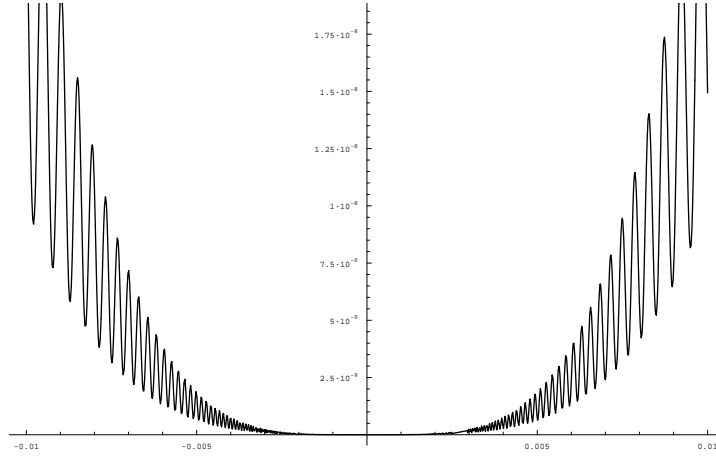


Figure 4:  $f(x) := x^4 \left(2 + \sin \frac{1}{x}\right)$

The function [2, p. 36]

$$f(x) := \begin{cases} x^4 \left(2 + \sin \frac{1}{x}\right) & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{for } x = 0 \end{cases}$$

shows that this is wrong (Figure 4). Obviously,  $f$  has a strict local minimum at 0, and  $f$  is even continuously differentiable on  $\mathbb{R}$  with  $f'(0) = 0$ , but  $f'$  assumes both positive and negative values in any neighbourhood of 0 as we can easily see from

$$f'(x) = x^2 \cdot \left(8x + 4x \sin \frac{1}{x} - \cos \frac{1}{x}\right) \quad \text{for } x \neq 0.$$

2. Usually, a *torsion point* of a function  $f : [a, b] \rightarrow \mathbb{R}$  is defined to be a point  $x_0 \in ]a, b[$  such that  $f$  is convex on one side of  $x_0$  (more precisely, in a certain interval  $]x_0 - \delta, x_0[$  resp.  $]x_0, x_0 + \delta[$ ) and concave on the other side. According to this definition also a function like

$$f(x) := \begin{cases} x^2 & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \geq 0 \end{cases} \quad (3.1)$$

has a torsion point at  $x = 0$ . One might also think of the following alternative definition: A function  $f : [a, b] \rightarrow \mathbb{R}$  has a torsion point at  $x_0 \in ]a, b[$  if  $f$  is differentiable at  $x_0$  and if the graph of  $f$  is above the tangent line  $x \mapsto f(x_0) + f'(x_0) \cdot (x - x_0)$  on one side of  $x_0$  and below this tangent line on the other side. This definition does not apply to the function in (3.2), but in the case of functions which are differentiable at  $x_0$ , it is more general than the first definition. For example, if we set

$$f(x) := \begin{cases} x^3 + x^2 \sin^2 \frac{1}{x} & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ x^3 - x^2 \sin^2 \frac{1}{x} & \text{for } x < 0, \end{cases} \quad (3.2)$$

then  $f$  is differentiable at  $x = 0$  with  $f'(0) = 0$ ,  $f(x) > 0$  for all  $x > 0$  and  $f(x) < 0$  for all  $x < 0$ , so  $f$  has a torsion point at  $x = 0$  (in the sense of the alternative definition). However,  $f$  is not convex and not concave in  $]0, \delta[$  for any  $\delta > 0$ , and the same holds in the intervals  $] - \delta, 0[$  (see Figure 5).

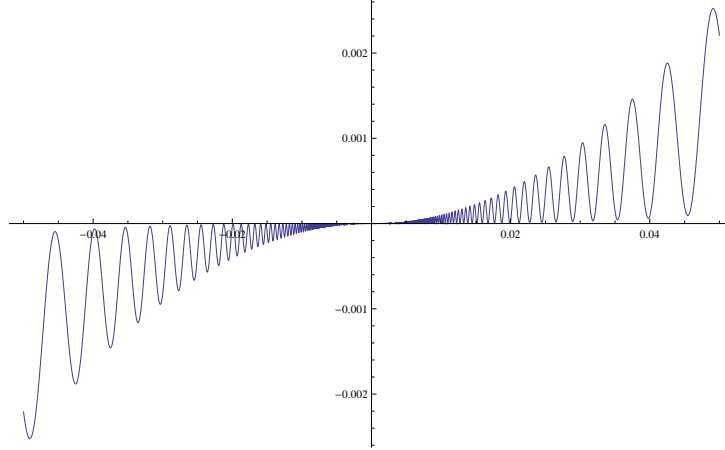


Figure 5:  $f(x) := x^3 + \text{sgn}(x) \cdot x^2 \sin^2 \frac{1}{x}$

This alternative definition of torsion point also applies to functions which are not continuous in any neighbourhood of the torsion point. For example, the function  $f : \mathbb{R} \longrightarrow \mathbb{R}$  defined by

$$f(x) := \begin{cases} x^3 & \text{for } x \in \mathbb{Q}, \\ x^5 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

has a torsion point in  $x = 0$ .

3. By Darboux's intermediate value theorem, if a derivative  $f'$  is not continuous at some  $x_0$ , then  $x_0$  has to be an oscillation point of  $f'$ . In the usual examples for this fact (like  $x \mapsto x^2 \sin \frac{1}{x}$ ),  $f'$  oscillates in both directions, above and below  $f'(x_0)$ , i.e. for arbitrary small neighbourhoods  $U$  of  $x_0$ ,  $f'(x_0)$  is an inner point of  $f'(U)$ .

The function

$$f(x) := \int_0^x \left| \cos \frac{1}{t} \right|^{1/|t|} dt$$

is an example of a differentiable function  $f : \mathbb{R} \longrightarrow \mathbb{R}$  which satisfies  $f'(0) = 0 \leq f'(x)$  for all  $x \in \mathbb{R}$  and whose derivative  $f'$  is not continuous in 0, i.e.  $f'$  oscillates only above  $f'(0)$ .

**Proof.** The function  $t \mapsto \left| \cos \frac{1}{t} \right|^{1/|t|}$  is bounded and continuous almost everywhere, so by Lebesgue's criterion it is integrable (even in the sense of Riemann) on compact intervals. Therefore,  $f$  is well-defined. For  $x \neq 0$  it is clear from the fundamental theorem of calculus that  $f'(x) = \left| \cos \frac{1}{x} \right|^{1/|x|} \geq 0$ .

So it remains to show that  $f'(0) = 0$ . For this purpose we use the substitution  $u := \frac{1}{t}$  and consider the points

$$a_k := k\pi, \quad b_k := \left(k + \frac{1}{k^{1/4}}\right) \cdot \pi, \quad c_k := \left(k + 1 - \frac{1}{k^{1/4}}\right) \cdot \pi \quad (k \in \mathbb{N}).$$

For  $k \geq 16$  we have  $a_k \leq b_k \leq c_k \leq a_{k+1}$ ,

$$\int_{a_k}^{b_k} \frac{|\cos u|^u}{u^2} du \leq \frac{b_k - a_k}{a_k^2} = \frac{1}{\pi \cdot k^{9/4}}$$

and

$$\int_{c_k}^{a_{k+1}} \frac{|\cos u|^u}{u^2} du \leq \frac{a_{k+1} - c_k}{c_k^2} \leq \frac{1}{\pi \cdot k^{9/4}}.$$

Furthermore, using the estimate  $\cos y \leq 1 - \frac{y^2}{4}$  for  $0 \leq y \leq \frac{\pi}{2}$ , for  $k \geq 16$  we obtain

$$\int_{b_k}^{c_k} \frac{|\cos u|^u}{u^2} du \leq \frac{c_k - b_k}{b_k^2} \cdot \left(\cos \frac{\pi}{k^{1/4}}\right)^{b_k} \leq \frac{1}{\pi k^2} \cdot \left(1 - \frac{\pi^2}{4\sqrt{k}}\right)^{k\pi}.$$

Here,  $\lim_{k \rightarrow \infty} \left(1 - \frac{\pi^2}{4\sqrt{k}}\right)^{\sqrt{k}\pi} = e^{-\pi^3/4} < 1$ , so if we set  $\alpha := \frac{1}{2} \cdot \left(1 + e^{-\pi^3/4}\right)$ , then there exists some  $N_0 \geq 16$  such that

$$\left(1 - \frac{\pi^2}{4\sqrt{k}}\right)^{\sqrt{k}\pi} \leq \alpha \quad \text{for all } k \geq N_0,$$

and we get

$$\int_{b_k}^{c_k} \frac{|\cos u|^u}{u^2} du \leq \frac{1}{\pi k^2} \cdot \alpha^{\sqrt{k}} \quad \text{for all } k \geq N_0.$$

Combining these estimates, for all  $N \geq N_0 - 1$  we deduce

$$\begin{aligned} 0 &\leq \int_0^{1/(N+1)\pi} \left|\cos \frac{1}{t}\right|^{1/|t|} dt = \int_{(N+1)\pi}^{\infty} \frac{|\cos u|^u}{u^2} du \\ &= \sum_{k=N+1}^{\infty} \left( \int_{a_k}^{b_k} \frac{|\cos u|^u}{u^2} du + \int_{b_k}^{c_k} \frac{|\cos u|^u}{u^2} du + \int_{c_k}^{a_{k+1}} \frac{|\cos u|^u}{u^2} du \right) \\ &\leq \sum_{k=N+1}^{\infty} \left( \frac{2}{\pi \cdot k^{9/4}} + \frac{1}{\pi k^2} \cdot \alpha^{\sqrt{k}} \right) \leq \frac{2}{\pi} \int_N^{\infty} \frac{dx}{x^{9/4}} + \frac{1}{\pi} \int_N^{\infty} \frac{\alpha^{\sqrt{x}}}{\sqrt{x}} dx \\ &= \frac{8}{5\pi} \cdot \frac{1}{N^{5/4}} + \frac{2}{\pi} \int_{\sqrt{N}}^{\infty} \alpha^y dy = \frac{8}{5\pi} \cdot \frac{1}{N^{5/4}} + \frac{2\alpha^{\sqrt{N}}}{\pi \log \frac{1}{\alpha}}. \end{aligned}$$

Now let some  $x > 0$  be given. W.l.o.g. we may assume  $x < \frac{1}{N_0\pi}$ . Then there exists some  $N_x \in \mathbb{N}$ ,  $N_x \geq N_0$  such that  $\frac{1}{(N_x+1)\pi} < x \leq \frac{1}{N_x\pi}$ , and we obtain

$$\begin{aligned} 0 &\leq \frac{f(x)}{x} \leq (N_x + 1)\pi \cdot \int_0^{1/N_x\pi} \left| \cos \frac{1}{t} \right|^{1/|t|} dt \\ &\leq \frac{8}{5} \cdot \frac{N_x + 1}{(N_x - 1)^{5/4}} + \frac{2}{\log \frac{1}{\alpha}} \cdot (N_x + 1) \alpha^{\sqrt{N_x-1}} \longrightarrow 0 \text{ for } N_x \rightarrow \infty. \end{aligned}$$

Since for  $x \rightarrow 0+$  we have  $N_x \rightarrow \infty$ , we can conclude that  $\frac{f(x)}{x} \rightarrow 0$  for  $x \rightarrow 0+$ . In view of  $f(-x) = -f(x)$  the same holds for  $x \rightarrow 0-$ . So  $f$  is differentiable at 0 with  $f'(0) = 0$ .

That  $f'$  is not continuous at 0 is clear. ■

This example can be easily modified such that  $f'$  is even unbounded near the origin, by setting

$$f(x) := \int_0^x \frac{1}{\sqrt{|t|}} \cdot \left| \cos \frac{1}{t} \right|^{1/|t|^2} dt.$$

Furthermore, functions with the properties discussed in this item can also be constructed as integral functions of piecewise linear functions, having increasingly smaller and increasingly higher peaks which accumulate at the origin.

4. There are certain analogies between infinite series and improper Riemann integrals of the form  $\int_0^\infty f(t) dt$ . For example, if  $f : [0, \infty[ \rightarrow [0, \infty[$  is monotonically decreasing, then the convergence of the improper integral  $\int_0^\infty f(t) dt$  is equivalent to the convergence of the series  $\sum_{n=0}^\infty f(n)$ .

Now if  $\sum_{n=0}^\infty a_n$  is a convergent series,  $(a_n)_n$  necessarily converges to 0. This might (mis)lead to the conjecture that if the improper Riemann integral  $\int_0^\infty f(t) dt$  converges (where  $f$  is a continuous function) then  $\lim_{t \rightarrow \infty} f(t) = 0$ . This conjecture is wrong. The function

$$f(x) := x \sin(x^3)$$

shows that on the contrary  $f$  might be even unbounded for  $x \rightarrow \infty$ . (To show the convergence of  $\int_0^\infty f(t) dt$  we substitute  $y = \varphi(t) := t^3$  which gives

$$\int_1^R f(t) dt = \int_1^{R^3} \frac{\sin y}{3y^{1/3}} dy \quad \text{for all } R > 0$$

and observe that the improper integral  $\int_1^\infty \frac{\sin y}{3y^{1/3}} dy$  converges.)

5. The function

$$f(x) := \frac{\sin(x^3)}{x}$$

satisfies  $\lim_{x \rightarrow \infty} f(x) = 0$ , but  $\lim_{x \rightarrow \infty} f'(x)$  does not exist, and  $f'$  is even unbounded near  $\infty$ . (In fact  $f'$  behaves very much like the function in 4.)

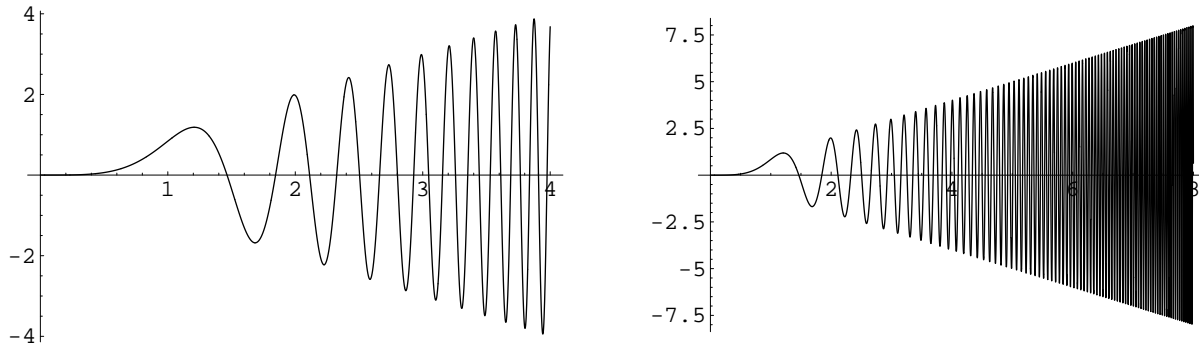


Figure 6:  $f(x) = x \sin(x^3)$  in the intervals  $[0; 4]$  and  $[0; 8]$

## 4 Extremal points of functions of several variables

When discussing extremal points of a real-valued  $C^2$ -function  $f$  of several (say  $n$ ) variables, the Hessian  $H_f$  of  $f$  is an inevitable tool. The fact that the definiteness or indefiniteness of  $H_f(\xi)$  at a critical point  $\xi$  provides information whether  $f$  has an extremum at  $\xi$  can be motivated in the following way: For an arbitrary vector  $v \in \mathbb{R}^n$  of unit length, the term  $v^T H_f(\xi) v$  can be considered as the second directional derivative of  $f$  at  $\xi$  along  $v$  (i.e. as the second derivative of  $t \mapsto g_v(t) := f(\xi + tv)$  at  $t = 0$ ). So if, for example,  $f$  has a minimum at  $\xi$  and hence for each direction  $v$  the function  $g_v$  has a minimum at  $t = 0$ , from a well-known criterion from one-dimensional analysis one gets that  $g_v''(0) \geq 0$ , i.e.

$$v^T H_f(\xi) \cdot v = \frac{\partial^2 f}{\partial v^2}(\xi) \geq 0 \quad \text{for all directions } v,$$

which means that  $H_f(\xi)$  is positive definite.

On the other hand, if  $H_f(\xi)$  is positive definite, we have

$$\frac{\partial^2 f}{\partial v^2}(\xi) = v^T H_f(\xi) \cdot v > 0 \quad \text{for all directions } v,$$

so for each direction  $v$ , the function  $g_v$  has a strict minimum at  $t = 0$ . More precisely (using the stronger fact that  $v \mapsto v^T H_f(\xi) \cdot v$  attains a *positive* minimum on the compact sphere  $S^{n-1}$ ) one can show that  $f$  itself has a strict minimum at  $\xi$ .

These considerations might lead to the following conjecture: If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and  $t \mapsto f(\xi + tv)$  (as a function  $\mathbb{R} \rightarrow \mathbb{R}$ ) has a strict local minimum in  $t = 0$  for every direction  $v \neq 0$ , then  $f$  itself has a local minimum at  $\xi$ .

This conjecture is false as the following example shows [2, p. 122] :

$$f(x, y) := (5y - x^2)(y - x^2) = 5y^2 - 6x^2y + x^4.$$

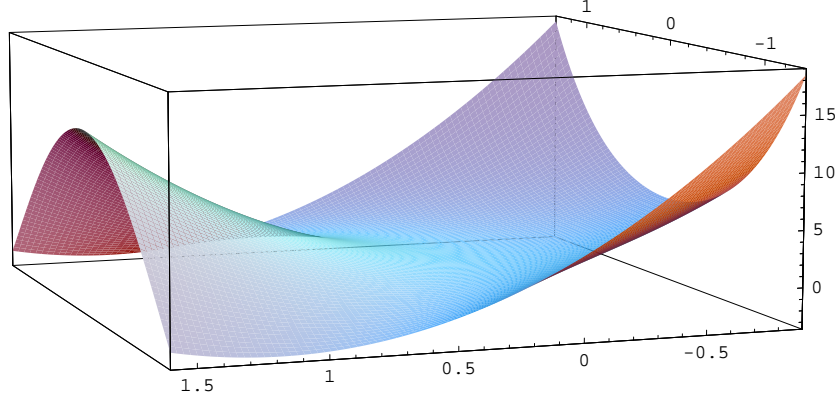


Figure 7: Graph of  $f(x, y) := (5y - x^2)(y - x^2)$ .

For all directions  $v \in \mathbb{R}^2$  and all  $t \in \mathbb{R}$  we have

$$f(tv) = f(tv_1, tv_2) = t^2 \cdot (5v_2^2 - 6v_1^2v_2t + v_1^4t^2).$$

If  $v_2 \neq 0$ , this shows immediately that  $t \mapsto f(tv)$  attains a strict local minimum at  $t = 0$ . If  $v_2 = 0$  we have  $v_1 \neq 0$  and

$$f(tv) = v_1^4 t^4,$$

so  $t \mapsto f(tv)$  attains a strict local minimum at  $t = 0$  as well. However, there are other ways to approach the origin than just on straight lines - for example on parabolae. And indeed, from

$$f(2t, t^2) = -3t^4$$

we see that  $f$  attains negative values in every neighbourhood of the origin, i.e. it cannot have a local minimum there.

This example has an easy geometrical explanation (see Figure 8):  $f(x, y)$  is negative iff  $\frac{1}{5}x^2 < y < x^2$ , hence in the area  $A$  between the parabolae  $y = \frac{1}{5}x^2$  and  $y = x^2$ . But for each line  $L$  through the origin there is some neighbourhood  $U$  of the origin such that  $L$  does not intersect  $A$  in  $U$ , hence  $f$  is non-negative on this part of  $L$ .

In this example,  $H_f(0)$  is positive semi-definite; by the compactness argument mentioned above, this phenomenon cannot occur if  $H_f(\xi)$  is positive definite.

We finish this paper with a somewhat crazier example, replacing the straight lines from the last example by an “oscillating curve”: Consider the functions  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$g(x, y) := \begin{cases} \exp\left(-\frac{1}{x^2} - \frac{1}{(y - \sin \frac{1}{x})^2}\right) & \text{if } x \neq 0 \text{ and } y > \sin \frac{1}{x}, \\ -\exp\left(-\frac{1}{x^2} - \frac{1}{(y - \sin \frac{1}{x})^2}\right) & \text{if } x \neq 0 \text{ and } y < \sin \frac{1}{x}, \\ 0 & \text{if } x = 0 \text{ or } y = \sin \frac{1}{x}. \end{cases}$$

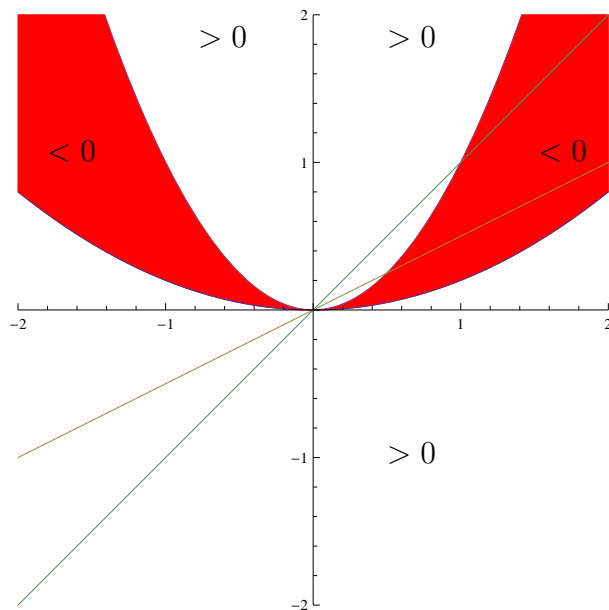


Figure 8: The region where  $f(x, y) = (5y - x^2)(y - x^2) = 5y^2 - 6x^2y + x^4$  is negative

and

$$f(x, y) := g(x, y) + \exp\left(-\frac{1}{x^4}\right).$$

Then  $f$  is a  $C^\infty$ -function on  $\mathbb{R}^2$  and  $f(x, \sin \frac{1}{x}) > f(0, 0) = 0$  for all  $x \in \mathbb{R}$ . So the restriction of  $f$  to the curve  $x \mapsto (x, \sin \frac{1}{x})$  has a strict minimum in the origin. However,  $f$  doesn't assume a local or global minimum at the origin since it assumes both positive and negative values in each neighbourhood of the origin.

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